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Kaon mixing and the charm mass

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Abstract: We study contributions to the $\Delta S = 2$ weak Chiral lagrangian producing K^0-$\overline{K}^0$ mixing which are not enhanced by the charm mass. For the real part, these contributions turn out to be related to the box diagram with up quarks but, unlike in perturbation theory, they do not vanish in the limit $m_u \to 0$. They increase the leading contribution to the $K_L - K_S$ mass difference by $\sim 10\%$. This means that short distances amount to $(90 \pm 15)\%$ of this mass difference. For the imaginary part, we find a correction to the $\lambda_c^{*2} m_c^2$ term of -5% from the integration of charm, which is a small contribution to ϵ_K. The calculation is done in the large-N_c limit and we show explicitly how to match short and long distances.

Keywords: 1/N Expansion, Standard Model, Kaon Physics, Chiral Lagrangians.

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Contents

1. Introduction

The Standard Model induces strangeness changing transitions by two units through the famous box diagram connecting a K^0 to a $\overline{K}^0$ first considered in the pioneering paper by Gaillard and Lee in 1974 [1]. This diagram is depicted in figure 1 where, as one can see, all quarks u, c and t are allowed to run in the box. The result of this diagram is

$$\mathcal{H}_{\text{eff}}^{S=2} \approx \frac{G_F^2}{4\pi^2} \overline{s}_L \gamma^\mu d_L(x) \overline{s}_L \gamma_\mu d_L(x) \times \\ \times c(\mu) \left\{ \eta_1 \lambda_c^2 m_c^2 + \eta_2 \lambda_t^2 \left(m_t^2\right)_{\text{eff}} + 2\eta_3 \lambda_c \lambda_t m_c^2 \log \frac{m_t^2}{m_c^2} \right\} + \text{h.c.}\,, \tag{1.1}$$

where we have defined $\psi_{L,R} = \frac{1 \mp \gamma_5}{2} \psi$, $\lambda_i = V_{is}^* V_{id}$ and $m_{u,c}$ stand for the corresponding quark masses.[1] In the case of the top we have defined the effective mass [7],

$$\left(m_t^2\right)_{\text{eff}} = 2.39 M_W^2 \left[\frac{m_t}{167\,\text{GeV}}\right]^{1.52}, \tag{1.2}$$

which takes into account that the top mass is heavier than the W. In the (unrealistic) limit $M_W \gg m_t \gg m_c$ one would have $(m_t^2)_{\text{eff}} \to m_t^2$. Physical observables such as the $K_L - K_S$ mass difference and ϵ_K get a contribution from the real and imaginary part, respectively, of the matrix element $\langle K^0 | \mathcal{H}_{\text{eff}}^{S=2} | \overline{K}^0 \rangle$.

As it is usually done, in eq. (1.1) we have neglected the up quark contribution to the box diagram because it comes with a m_u^2 factor out front. Therefore, one could naively conclude that the matrix element $\langle K^0 | \mathcal{H}_{\text{eff}}^{S=2} | \overline{K}^0 \rangle$ is given by the operator (1.1) up to very small corrections $\mathcal{O}(m_u^2/m_c^2)$.

[1]For the meaning of $c(\mu)$ and η_i, see below.

However, this argument is flawed because the above box-diagram was obtained assuming that strong interactions are purely perturbative, which is obviously not true. Strong interactions bind light quarks very tightly and produce a mass gap between the (π, K, η) octet and and the first hadronic resonance. Consequently, relative to the kaon mass, there are two kinds of contributions to the K^0-$\overline{K}^0$ transition depending on the hadronic scale governing the intermediate states. On the one hand, there is the contribution in which the intermediate states are members of the Goldstone octet. This contribution is given by the effect of the $\Delta S = 1$ chiral lagrangian acting twice and is nonlocal at the scale of the kaon mass. In a chiral expansion, the leading term in this contribution is of $\mathcal{O}(p^2)$ but vanishes due to the Gell-Mann-Okubo mass relation. The remaining subleading term at $\mathcal{O}(p^4)$, regretfully, depends on unknown low-energy coupling constants. An estimate of this term was done in ref. [2] but the result depends on a cutoff and the uncertainties are very large. It is clear, therefore, that a reliable calculation of these $(\Delta S = 1) \times (\Delta S = 1)$ contributions remains a very important issue. In this work, however, we shall not deal with it.

On the other hand, there is the contribution in which all the intermediate states are (much) heavier than the kaon. Among these there are of course the c, b, t quarks and the W boson, but *also* all the hadronic resonances with u, d, s quark content. This second type of contributions to the K^0-$\overline{K}^0$ transition gives rise to a $\Delta S = 2$ chiral lagrangian. The above box diagram with an up quark is a misrepresentation for the contribution coming from these resonances.

In the case of the contributions from c, b, t one can of course include gluon corrections in a perturbative way, and this has been done in refs. [3, 4, 5]. These perturbative corrections can be lumped into the $c(\mu)$ and η_i coefficients, with the result [7]

$$\begin{aligned}
\eta_1 &= (1.32 \pm 0.32) \left[\frac{1.3\,\mathrm{GeV}}{m_c(m_c)}\right]^{1.1}, \qquad \eta_2 = 0.57 \pm 0.01\,, \qquad \eta_3 = 0.47 \pm 0.05 \\
c(\mu) &= (\alpha_s(\mu))^{-\frac{9}{11 N_c}} \left[1 + \frac{\alpha_s(\mu)}{\pi}\left(\frac{1433}{1936} + \frac{1}{8}\kappa\right)\right], \qquad (1.3)
\end{aligned}$$

where $\kappa = 0$ (-4) in the naive dimensional regularization (resp. 't Hooft-Veltman) schemes.

Obviously, in the case of the up quark contribution in eq. (1.1) there is no point in including any perturbative gluon correction to it. Furthermore, since the up quark contribution is proportional to λ_u^2, which is real, it can only contribute to the $K_L - K_S$ mass difference. Then, the following natural question arises: since m_u^2 is misleading, what is the true size of this up quark contribution to the $K_L - K_S$ mass difference? As we have argued above, there is a nonperturbative contribution which comes from integrating out hadronic resonances. Therefore, in principle, there could be a contribution proportional to a typical hadronic scale $\Lambda_{\mathrm{QCD}} \sim 1\,\mathrm{GeV}$, so that the corrections to the $K_L - K_S$ mass difference would become $\mathcal{O}(\Lambda_{\mathrm{QCD}}^2/m_c^2)$ and could be very important.

Furthermore, since $\lambda_{c,t}$ have a large imaginary part, there is an important contribution to the ϵ_K parameter in the terms proportional to m_c^2 in eq. (1.1). Could hadronic effects proportional to Λ_{QCD}^2 produce large corrections to, e.g., the imaginary part of the $\lambda_c^2 m_c^2$ term in (1.1)?

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As a matter of fact, both questions above are related because they require considering the problem of subleading corrections to the large-m_c expansion. Somewhat surprisingly, not much work has been devoted to the study of $1/m_c$ corrections. Apart from the problem of K^0-$\overline{K}^0$ mixing, we are only aware of refs. [8, 9, 10, 11] where these corrections have been considered. In the specific case of the K^0-$\overline{K}^0$ matrix element, Bijnens, Gerard and Klein made an estimate in ref. [12]. They restricted themselves to the contribution coming solely from Goldstone loops, selecting the leading topologies in the large-N_c limit, and using a sharp cutoff to regulate the ultraviolet divergences which resulted. In this way they obtained a positive correction to the quadratic charm mass contribution to the $K_L - K_S$ mass difference of the form $\lambda_c^2(m_c^2 + \Lambda_H^2)$, with $\Lambda_H \sim 0.5\,\mathrm{GeV}$. Although there was no matching between short and long distances and, as a consequence, the physical result for the K^0-$\overline{K}^0$ matrix element was cutoff dependent, there was reasonable numerical stability against variations of the cutoff in the neighborhood of this value for Λ_H.

Pivovarov [13] considered the OPE and obtained matching at the level of quark diagrams (in $\overline{MS}$) between the contribution to the box diagram with intermediate charm and the contribution with only up quarks. However his calculation did not show how to implement matching with mesons and, in particular, did not include the constraints coming from chiral symmetry in the low-energy region. As a consequence, his result was very sensitive to an ad-hoc infrared cutoff. Relative to the $\lambda_c^2 m_c^2$ contribution to the box diagram, he obtained a correction in the range $-0.4 \lesssim \pm\Lambda_H^2/m_c^2 \lesssim +0.1$, depending on the choice for this infrared regulator.

It seems to us, therefore, that understanding the size of these corrections is an interesting issue. In this work we shall reconsider the calculations of refs. [12, 13] in a framework which incorporates the correct infrared and ultraviolet behavior and which allows matching between long and short distances. As we shall see, the crucial ingredient for obtaining this matching is the consideration of the constraints imposed by the Operator Product Expansion (OPE).

Let us back up a bit and describe how to organize the calculation using Effective Field Theory. In an Effective Field Theory calculation one integrates out all fields whose masses are larger than the kaon mass. This means that one has a theory in which the quarks c, b, t, the gauge bosons W, Z and the Higgs are no longer explicit degrees of freedom in the lagrangian. Only the u, d, s quarks and the gluons propagate. A set of matching conditions ensures that, even though the ultraviolet properties of the two lagrangians are very different, the physics from the Effective lagrangian does not differ from the one of the original Standard Model. The advantage of this way of organizing the calculation is that one disentagles high- from low-energy scales, one at a time, and can treat first all the heavy particles in a perturbative way, order by order in powers of the coupling α_s. This perturbative running from the W scale down to the charm scale is very well understood and under very good theoretical control thanks, among others, to the work of refs. [3, 4, 6, 5].

After the integration of charm one still has a lagrangian in terms of the *quarks* u, d, s, not the associated mesons, so that the calculation of matrix elements between kaon states is still a nontrivial task. However, kaons are pseudo-Goldstone bosons and therefore their interactions can be organized in a Chiral lagrangian in powers of derivatives and masses.

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This means that the quark operators obtained in the previous perturbative running can be bosonized in terms of kaon fields as explicit degrees of freedom. Once this is done, the calculation of a kaon matrix element is straightforward. But in the construction of this chiral lagrangian, exactly as before, one has to make sure that a set of matching conditions are satisfied. This point has been recognized only recently [14, 15]. These conditions are no longer perturbative in powers of α_s and require a nonperturbative treatment; with the help of the OPE, they control the interplay between long and short distances.

Obviously these matching conditions cannot be solved exactly and one is forced to do some approximations. Since, as usual, these conditions equate a Green's function computed in terms of quarks to the same Green's function written in terms of mesons, it is very convenient to use an approximation which can be carried out for both degrees of freedom. The approximation we will use for this matching is the large-N_c expansion in QCD [16, 17].

The above program has been applied to other problems as well, including how to estimate low-energy coupling constants in quenched QCD [19]. We refer to ref. [18] for a list of references.

In section 2 we shall review a toy version of the calculation done by Gilman and Wise to illustrate how the perturbative running is done, keeping only dimension-six operators. Section 3 will be devoted to the inclusion in this perturbative running of dimension-eight operators, suppressed by $1/m_c^2$ relative to the former, after the integration of charm. Section 4 will contain the integration of resonances with the corresponding matching to long distances. Finally, our conclusions will appear in section 5.

2. The Gilman-Wise calculation: simplified version

Let us imagine[2] that the top were as heavy as the W, i.e. $m_t \sim M_W \sim \mu_W$. Then, at the scale $\mu = \mu_W$, the box diagram in figure 1 would give rise to a local effective operator given by

$$\mathcal{L}_{\mathrm{eff}} = \frac{G_F^2}{16\pi^2} c(\mu) \left\{ \overline{s}\gamma^\mu (1-\gamma_5) d(x) \right\}^2 . \tag{2.1}$$

What this means is that shrinking the W propagator to a local Fermi interaction and leaving out the top contribution differs from the full result in figure 1 by an amount that can be encoded in the operator (2.1), with the value of the coefficient given by $c(m_t) = -m_t^2 \lambda_t^2$. This is the matching condition for the coefficient $c(\mu)$ in the theory without top quarks and W bosons at $\mu = \mu_W$. This value of $c(m_t)$ is to be taken as the boundary condition in the differential equations controlling the running down to scales $\mu \ll \mu_W$.

In the region $m_c < \mu < \mu_W$, the running of the operator (2.1) is given by the $1/\varepsilon$ poles of the diagrams in figure 2 in dimensional regularization, where only the quarks c and u run around the loop. This leads to the following differential equation

$$\mu^2 \frac{d}{d\mu^2}\, c(\mu) = -2m_c^2 \lambda_c \left(\lambda_c + \lambda_u\right) - 2m_u^2 \lambda_u \left(\lambda_c + \lambda_u\right) . \tag{2.2}$$

[2] This discussion parallels that of ref. [20].

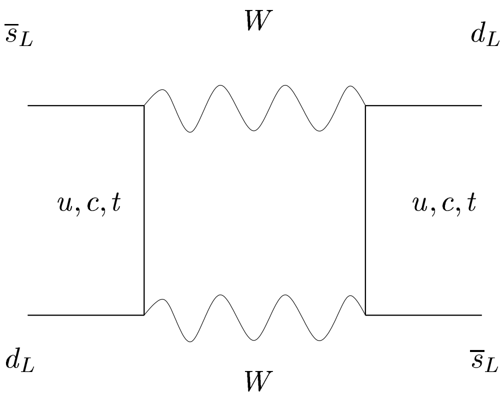

Figure 1: Box diagram giving rise to $\Delta S = 2$ transitions in the Standard Model.

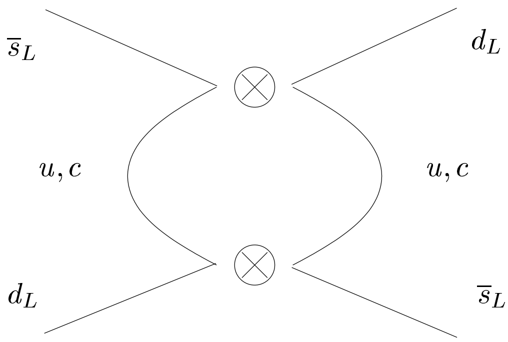

Figure 2: Diagram contributing to the running of the operator (2.1) in the Effective Theory below $\mu_W \sim m_t \sim M_W$.

Because in this calculation we are doing perturbation theory, the parameter m_u is just the up quark mass and its contribution may be safely neglected. With the above boundary condition, this equation can be trivially integrated to (setting $m_u = 0$)

$$c(\mu) = -m_t^2 \lambda_t^2 + 2m_c^2 \lambda_c (\lambda_c + \lambda_u) \log \frac{m_t^2}{\mu^2} \,. \tag{2.3}$$

At $\mu = m_c$, one has to integrate out the charm.[3] The diagram in figure 2 with only the c quark running in the loop requires the coefficient in the operator (2.1) in this new theory *without* charm to satisfy the matching condition $c(m_c) = -m_c^2 \lambda_c^2$, just like the condition from the integration of the top. Therefore, at μ just below m_c one has a theory with only the quarks u, d, s and with the coefficient of the operator (2.1) given by

$$c(\mu \lesssim m_c) = -m_t^2 \lambda_t^2 + 2m_c^2 \lambda_c (\lambda_c + \lambda_u) \log \frac{m_t^2}{m_c^2} - m_c^2 \lambda_c^2 \,. \tag{2.4}$$

At $\mu < m_c$ the perturbative running of $c(\mu)$ is given by the diagram of figure 2 with solely the up quark running in the loop. However, this contribution is proportional to m_u. Setting $m_u = 0$, the operator (2.1) no longer runs. Therefore, the coefficient $c(\mu)$ "freezes out" at a scale μ right below the charm mass, i.e. at a scale which can still be considered perturbative, and it is given by eq. (2.4).

Of course our calculation has been too simplistic in two respects. Firstly, m_t is larger than M_W and, secondly, we neglected α_s corrections in the whole discussion of running and matching. The modifications to amend this are nontrivial and were done in refs. [3, 4, 6, 5]. They give rise to the coefficients $\eta_{1,2,3}$ and the effective mass $(m_t^2)_{\text{eff}}$ appearing in eq. (1.1) (after use of the unitarity condition $\lambda_u + \lambda_c = -\lambda_t$ has been made). Furthermore, α_s corrections "defrost" the coefficient $c(\mu)$ below m_c and make it run in the region $M_\rho \lesssim \mu \lesssim m_c$ like in eq. (1.3), where it matches to the weak chiral lagrangian coming from long distances [21]. However, and most importantly, all these short-distance modifications are always proportional to the quark masses m_t or m_c and do not change the conclusion of our

[3]In this simplified version the quark bottom does not play any role.

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simplified analysis: at this level there are no power corrections of the scale $\Lambda_{\rm QCD} \sim 1\,{\rm GeV}$. For these corrections to show up, one needs to go to dimension-eight operators. This will be the subject of the next section.

3. Beyond Gilman-Wise: dimension-eight operators

In this section we shall consider the $\Delta S = 2$ dimension-eight operators which are generated by the successive integration of the top, the W, and finally charm. Schematically, these operators are of the form

$$\mathcal{L}^{\rm dim\ 8}_{\Delta S=2} \sim G_F^2 \overline{s}_L \widehat{\rm A} d_L \overline{s}_L \widehat{\rm B} d_L \,, \tag{3.1}$$

with $(\widehat{\rm A}; \widehat{\rm B})$ a gluon field strength $(G_{\mu\nu}; 1)$ or two covariant derivatives $(D_\mu; D_\nu)$ acting on the quark fields.

In principle, the proliferation of scales and operators makes this a complicated problem. To simplify matters, we shall neglect logarithmic corrections such as $\log(m_t/M_W)$ in front of larger logarithms such as $\log(m_t/m_c)$. This means that we may integrate out simultaneously the top and the W at a common scale, $\mu_W \sim m_t \sim M_W$.

At the scale μ_W, $\Delta S = 2$ dimension-eight operators may get generated from the difference between the diagrams in figure 1 and their counterparts in the Effective Theory without the top and the W boson. The different contributions can be broken down as follows. Firstly, the top contribution will give terms proportional to λ_t^2; and the mixed diagrams involving the top and the up and charm quarks will give terms proportional to $\lambda_t(\lambda_c + \lambda_u) = -\lambda_t^2$, because of unitarity. Notice that at the matching scale μ_W the u, c quarks are effectively massless. Moreover, the diagrams with only the u, c quarks in the loop will be proportional to $(\lambda_c + \lambda_u)^2 = \lambda_t^2$. Therefore, the net result at μ_W is that all contributions are modulated by the factor $\lambda_t^2 \sim \mathcal{O}(10^{-8})$ which is tiny and can be neglected. Notice that the λ_t^2 term in eq. (1.1) is kept only because it is accompanied by the large $(m_t)^2_{\rm eff}$ factor. This factor now is impossible because, by dimensional analysis, there can be no m_t in front of the operator (3.1).

However, at the scale μ_W there are $\Delta S = 1$ dimension-six operators which come from W exchange at $\mathcal{O}(G_F)$. The flavor structure of these $\Delta S = 1$ operators is given schematically by

$$\mathcal{L}^{\rm dim\ 6}_{\Delta S=1} \sim G_F \sum_{x,y=u,c} V^*_{xs} V_{yd} \overline{s}_L x_L \overline{y}_L d_L \,, \tag{3.2}$$

with any combination of color indices.

One should now run down to the scale $\mu = m_c$. In so doing, squaring one of the operators (3.2) will produce a $\Delta S = 2$ dimension-eight operator, such as (3.1). However, unlike in the dimension-six case (2.2), there can be no mass factors in eq. (3.1). Therefore, not only in the matching condition at μ_W but also in the logarithmic running, the c and u quarks are effectively degenerate. Furthermore, in the mixing of $\mathcal{L}^{\rm dim\ 6}_{\Delta S=1} \times \mathcal{L}^{\rm dim\ 6}_{\Delta S=1}$ into the operator (3.1) the x, y indices in (3.2) have to be contracted. Thus, this mixing arranges the GIM cancelation in the sum

$$\left(\sum_{x=u,c} V_{xs}^{*} V_{xd}\right)^{2}, \tag{3.3}$$

and equals $(V_{ts}^{*}V_{td})^2 = \lambda_t^2$, again. Since this is a consequence of flavor symmetry, gluons cannot alter this conclusion.

At the scale $\mu = m_c$, charm gets integrated out. At this point we bring in the large-N_c approximation, which we shall use in the rest of this work. In the large-N_c limit, four-quark operators factorize and become a product of two independent color singlets. This is true even in the presence of gluon operators. Furthermore, we should keep in mind that we are eventually interested in matrix elements between a K^0 and a $\overline{K}^0$, to lowest order in the chiral expansion, i.e. $\mathcal{O}(p^2)$. There are two ways in which we can get a dimension-eight operator: either as a product of two dimension-four operators, or as a product of a dimension-3 operator times a dimension-5 one. The first case yields contributions of higher chiral order, namely $\mathcal{O}(p^4)$. This is because

$$\langle \overline{K}^0(p)|\bar{s}_L\gamma_\nu \mathcal{D}_\mu d_L|0\rangle \sim \left(p_\mu p_\nu - \frac{p^2}{4} g_{\mu\nu}\right), \tag{3.4}$$

as can be immediately seen by contracting with $g^{\mu\nu}$ and use of the equations of motion in the chiral limit. So, only the possibility $\mathcal{O}(\dim - 3)\times\mathcal{O}(\dim - 5)$ is left. However, $\overline{s}\widetilde{G}^{\mu\nu}\gamma_\nu d(x)$ is the *only* dimension-five current connecting a kaon to the vacuum [22]. Remarkably, this implies that there is only one $\Delta S = 2$ dimension-eight operator whose matrix element is of $\mathcal{O}(p^2)$ between a K^0 and a $\overline{K}^0$ with momentum p. This operator is given by

$$\mathcal{L}_{\rm eff} = -\frac{G_F^2}{3\pi^2} c_8(\mu)\mathcal{O}_8(\mu) + \text{h.c.} \qquad \text{where } \mathcal{O}_8(x) = g_s \overline{s}_L \widetilde{G}^{\mu\nu}\gamma_\nu d_L(x)\overline{s}_L\gamma_\mu d_L(x)\,, \tag{3.5}$$

where g_s is the strong coupling constant. This operator (3.5) is CPS symmetric, where CPS symmetry [26] is the combination of charge conjugation, parity and the exchange of quarks $s \leftrightarrow d$.[4]

At the scale $\mu = m_c$, the diagram in figure 3 with charm and up quarks running in the loop yields for the matching condition[5]

$$c_8(m_c) = -\frac{7}{6}\lambda_u^2 - \frac{13}{6}\,\lambda_c\lambda_u\,. \tag{3.6}$$

The term in eq. (3.6) proportional to λ_u^2 is, of course, purely real. However, the term proportional to $\lambda_c\lambda_u$ contains an imaginary part which means that the parameter ϵ_K receives a contribution from the operator $\mathcal{O}_8$ at the charm scale. Below m_c only the up quark is active and, therefore, the running of the operator $\mathcal{O}_8$ is proportional to λ_u^2, and purely real. Consequently, we find that there are no further corrections to ϵ_K coming from scales $\mu \sim \Lambda_{\rm QCD} \sim 1\,\text{GeV}$ from the integration of resonances.

[4] We have found that the operator basis used in ref. [13] does not respect this CPS symmetry, the offending operator being $\overline{s}_L\gamma_{\{\mu}D_{\nu\}}d_L\overline{s}_L\gamma^{\{\mu}D^{\nu\}}d_L$, where $\{,\}$ means symmetrization. However, this operator has vanishing matrix elements between a K^0 and a $\overline{K}^0$ in the large-N_c limit and, therefore, drops out of our analysis.

[5] The approximate relation $\lambda_c^2 + 2\lambda_c\lambda_u \simeq -\lambda_u^2$ has been used.

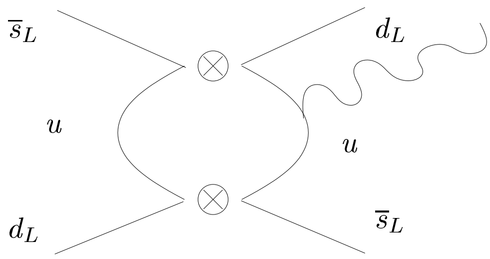

Figure 3: Diagram responsible for the running of the operator (3.5) proportional to λ_u^2.

Now, in the large-N_c limit, gluon corrections and the contribution from the diagram in figure 3 makes the operator (3.5) run according to (see appendix A)

$$\mu^2 \frac{d}{d\mu^2} c_8(\mu) = \lambda_u^2 + \gamma_8 \alpha_s c_8(\mu) \,, \tag{3.7}$$

where

$$\gamma_8 = \frac{4}{6\pi}\left(N_c - \frac{1}{N_c}\right) \approx \frac{4}{6\pi} N_c \,, \tag{3.8}$$

was first computed in ref. [27]. Now, one should integrate eq. (3.7) in the region $\mu \lesssim m_c$ subject to the boundary condition for $c_8(m_c)$ in eq. (3.6). To a sufficient approximation, one can actually neglect the α_s term and obtain the rather simple result

$$c_8(\mu) = -\frac{13}{6}\lambda_c\lambda_u + \lambda_u^2\left(\log\frac{\mu^2}{m_c^2} - \frac{7}{6}\right). \tag{3.9}$$

Therefore, at scales $\mu \lesssim m_c$, the Effective Field Theory is given by

$$\mathcal{L}_{\rm eff} = -\frac{G_F^2}{3\pi^2} c_8(\mu) g_s \overline{s}_L \widetilde{G}^{\mu\nu}\gamma_\nu d_L \overline{s}_L \gamma_\mu d_L - \frac{G_F}{\sqrt{2}}\lambda_u \overline{s}_L \gamma^\mu u_L \overline{u}_L \gamma_\mu d_L + \text{h.c.}\,, \tag{3.10}$$

where the first (second) operator produces $\Delta S = 2$ (respectively $\Delta S = 1$) transitions. This second operator is none other than Q_2. In eq. (3.10) the Wilson coefficient of Q_2, z_2, has been set to unity and no other $\Delta S = 1$ operator is considered. This is in accordance with the large-N_c limit. In real life, however, one has that $z_2(\mu \lesssim m_c) \simeq 1.3-1.4$ [6]. Subleading $1/N_c$ corrections are also naively expected to be of this size.[6]

At scales $\mu \sim M_K \ll m_c$ perturbation theory ceases to be valid. One has to change the description in terms of quarks and gluons in favor of meson fields as explicit degrees of freedom. This will be the subject of the next section.

4. Long-distance contribution

Below the charm mass the Effective Theory is given by the lagrangian (3.10) and is written in terms of the quarks u, d, s and gluons as explicit degrees of freedom.

[6]This naive expectation may turn out to be misleading when the leading term in $1/N_c$ is dominated by a chiral scale such as F_π [25]. This is not the case here.

In the large-N_c limit, however, and in terms of mesons as degrees of freedom, this Effective Theory contains, besides the octet of Goldstone bosons,[7] an infinite tower of resonances. In order to match to the lagrangian describing the only-Goldstone degrees of freedom, one must integrate out all the resonance fields.

At the scale of the kaon, there is only one $\mathcal{O}(p^2)$ $\Delta S = 2$ operator in the weak chiral lagrangian. It is given by

$$\mathcal{L}_{\rm eff}^{S=2} = \frac{G_F^2}{16\pi^2} F_0^4 \Lambda_{S=2}^2 \,\mathrm{Tr}\left[\lambda_{32}\left(D^\mu U^\dagger\right) U \lambda_{32}\left(D_\mu U^\dagger\right) U\right] + \mathrm{h.c.}\,, \tag{4.1}$$

where $[\lambda_{32}]_{ij} = \delta_{i3}\delta_{2j}$ is a (spurion) matrix in flavor space, $F_0 \simeq 0.087\,\mathrm{GeV}$ is the pion decay constant (in the chiral limit) and U is a 3×3 unitary matrix collecting the Goldstone boson degrees of freedom and transforming as $U \to RUL^\dagger$ under a flavor rotation (R,L) of the group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. In eq. (4.1) we have normalized with a dimensionful coupling constant $\Lambda_{S=2}$. The bosonization of the dimension-six operator obtained below the charm scale (section 2) yields for this coupling:

$$\Lambda_{S=2}^2|_{\rm dim-6} = g_{S=2}\left[\eta_1\lambda_c^2 m_c^2 + \eta_2\lambda_t^2\left(m_t^2\right)_{\rm eff} + 2\eta_3\lambda_c\lambda_t m_c^2 \log\frac{m_t^2}{m_c^2}\right], \tag{4.2}$$

where

$$g_{S=2} = 1 + \mathcal{O}\left(\frac{1}{N_c}\right) \tag{4.3}$$

is dimensionless, and the $\mathcal{O}(1/N_c)$ corrections have been computed in ref. [21].

We shall now consider the contribution to $\Lambda_{S=2}^2$ due to the dimension-eight operator obtained in the previous section. This is tantamount to matching the two lagrangians (3.10) and (4.1). In order to do this it is convenient to choose a Green's function which does not contain any contribution from the singlet combination $1_L \times 1_R$ under the flavor group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ [25]. This makes it explicit that the short distance properties be encoded in the Wilson coefficients of the standard four-quark effective operators and makes the following OPE analysis much simpler. As was done in refs. [21] we select the Green's function:

$$\mathcal{G}_{\alpha\beta}^{S=2}(p) = \int d^4x e^{ipx}\langle 0|\mathrm{T}\left\{R_\alpha^{ds}(x)R_\beta^{ds}(0)\right\}|0\rangle\,, \tag{4.4}$$

where $R_\alpha^{ds}(x) = \overline{d}_R\gamma_\alpha s_R(x)$.

The Effective lagrangian (4.1) yields for the Green's function $\mathcal{G}_{\alpha\beta}^{S=2}(p)$ at small momentum p,

$$\mathcal{G}_{\alpha\beta}^{S=2}(p) = -i\frac{G_F^2}{8\pi^2}F_0^4\Lambda_{S=2}^2\left(\frac{p_\alpha p_\beta}{p^2} - g_{\alpha\beta}\right) + \mathcal{O}(p^2)\,. \tag{4.5}$$

In terms of the lagrangian (3.10), the function $\mathcal{G}_{\alpha\beta}^{S=2}(p)$ is given by

$$\begin{aligned}\mathcal{G}_{\alpha\beta}^{S=2}(p) = &-i\frac{G_F^2}{3\pi^2}c_8(\mu)\int d^4x d^4z e^{ipx}\langle 0|\mathrm{T}\left\{R_\alpha^{ds}(x)\mathcal{O}_8(z)R_\beta^{ds}(0)\right\}|0\rangle - \\ &- i\frac{G_F^2}{2\pi^2}\lambda_u^2(4\pi\mu^2)^{\epsilon/2}\int dQ^2(Q^2)^{1-\epsilon/2}\int d\Omega_q\Gamma_{\mu\nu\alpha}(q,p)\Gamma^{\nu\mu\beta}(q,p)\end{aligned} \tag{4.6}$$

[7] In the large-N_c limit one should, strictly speaking, consider a nonet by including the η' [28].

where an expansion for low momentum p is understood, and only terms $\mathcal{O}(p^0)$ are retained. In eq. (4.6) we have defined the following Green's function

$$\Gamma_{\mu\nu\alpha}(q,p)=\lim_{p\to 0}\int d^4x d^4y e^{-i(qx+py)}\langle 0|\widehat{\mathrm{T}}\left\{R^{ds}_{\alpha}(y)L^{su}_{\mu}(x)L^{ud}_{\nu}(0)\right\}|0\rangle\,, \tag{4.7}$$

with $Q^2=-q^2$ being positive in the euclidean regime, $L^{su}_{\mu}(x)=\overline{s}_L\gamma_\mu u_L(x)$, etc., and where $d\Omega_q$ stands for the solid angle integration in $D=4-\epsilon$ dimensions, normalized so that

$$\int d\Omega_q q_\mu q_\nu=\frac{q^2}{D}\,g_{\mu\nu}\,. \tag{4.8}$$

The second term in (4.6) is the result of a double insertion of the operator Q_2 (see eq. (3.10)). In this second term an interesting subtlety arises. Because of the triangle anomaly, defining the function $\Gamma_{\mu\nu\alpha}(q,p)$ in eq. (4.7) in terms of the ordinary covariant T product introduces contributions which transform as $1_L\times 1_R$ under the flavor group $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$.[8] Since the lagrangian we want to match to in eq. (4.1) has an explicit $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ flavor symmetry,[9] it is clear that using a Green's function which violates this symmetry because of the anomaly is an unnecessary complication one would like to avoid. This is the reason why the Green's function $\Gamma_{\mu\nu\alpha}(q,p)$ in eq. (4.7) is defined in terms of a special T product, "$\widehat{\mathrm{T}}$". This $\widehat{\mathrm{T}}$ product is the one which produces the factorized, or left-right symmetric, form of the anomaly [23] and secures that the Green's function $\Gamma_{\mu\nu\alpha}(q,p)$ is actually anomaly free, satisfying naive Ward identities [24](see appendix B).

In the large-N_c limit we can use factorization and rewrite the first term in eq. (4.6) containing the operator $\mathcal{O}_8$ as

$$i\frac{G_F^2}{6\pi^2}c_8(\mu)\left(\frac{p_\alpha p_\beta}{p^2}-g_{\alpha\beta}\right)F_0^4\delta_K^2\,, \tag{4.9}$$

where, following ref. [22], we have defined a parameter δ_K^2 as

$$\langle 0|g_s\overline{s}_L\widetilde{G}^a_{\mu\nu}\lambda_a\gamma^\mu d_L|K^0(p)\rangle=-i\sqrt{2}F_0\delta_K^2 p_\nu\,. \tag{4.10}$$

In the equation above, λ_a are the color Gell-Mann matrices, normalized so that $\mathrm{Tr}\,\lambda_a\lambda_b=2\delta_{ab}$. We shall next deal with the second term in eq. (4.6).

As we discuss in appendix B, Ward identities restrict the form of the function $\Gamma_{\mu\nu\alpha}(q,p)$ in the low-p limit to be

$$\begin{aligned}\Gamma_{\mu\nu\alpha}(q,p)=&\frac{F_0^2}{2p^2q^2}\left[p^2q_\mu g_{\alpha\nu}+p^2q_\nu g_{\alpha\mu}-p_\alpha p_\beta q_\mu-p_\alpha p_\mu q_\nu+q_\mu q_\nu\left(\frac{p\cdot q}{q^2}p_\alpha-\frac{p^2}{q^2}q_\alpha\right)\right]+\\&+I_1(Q^2)\left(q^2g_{\mu\nu}-q_\mu q_\nu\right)\left(\frac{p\cdot q}{p^2}p_\alpha-q_\alpha\right)+\\&+I_2(Q^2)\left[i\varepsilon_{\mu\nu\lambda\sigma}q^\sigma\frac{p_\alpha p^\lambda}{p^2}-i\varepsilon_{\mu\nu\alpha\lambda}q^\lambda\right]+\mathcal{O}(p)\,,\end{aligned} \tag{4.11}$$

[8] After all, the anomaly can be computed in perturbation theory.

[9] Notice, in particular, that the result in eq. (4.5) is tranverse.

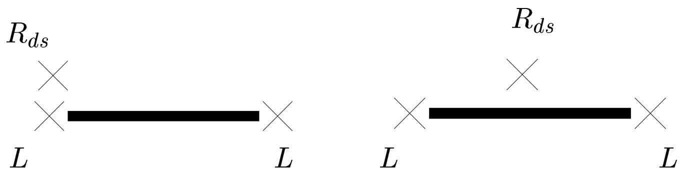

Figure 4: Pole structure of the function $\Gamma_{\mu\nu\alpha}$ in eq. (4.7). "L" stands for L_{ud} or L_{su}.

where $I_{1,2}(Q^2)$ are some unknown functions. The combination appearing in eq. (4.6), however, shows a much simpler tensor structure, to wit

$$\int d\Omega_q \Gamma_{\mu\nu\alpha}(q,p)\Gamma^{\nu\mu\beta}(q,p) = \left(\frac{p_\alpha p_\beta}{p^2} - g_{\alpha\beta}\right) W(Q^2) + \mathcal{O}(p^2)\,, \tag{4.12}$$

where

$$W(Q^2) = \frac{3}{4} I_1^2(Q^2) Q^6 - \frac{3}{2} I_2^2(Q^2) Q^2 + \frac{7}{16}\frac{F_0^4}{Q^2}\,. \tag{4.13}$$

The integral over Q^2 appearing in the matching condition (4.6) requires knowledge of the function $W(Q^2)$ for *all* values of Q^2 and this is not available. This is why computing electroweak matrix elements is difficult. However, we know how the function $W(Q^2)$ behaves at low values of Q^2 because this is given by Chiral Perturbation Theory. A straightforward, though somewhat lengthy, calculation leads to (one can take $\epsilon = 0$ here)

$$Q^2 W(Q^2) \approx \frac{7}{16} F_0^4 - \left[\frac{3}{2}\left(\frac{N_c}{24\pi^2}\right)^2 - 3L_9^2\right] Q^4 + \mathcal{O}(Q^6)\,, \tag{4.14}$$

where the first contribution in squared brackets is due to the Wess-Zumino term, with the anomaly subtracted to conform with the definition of $\Gamma_{\mu\nu\alpha}(q,p)$ and the $\widehat{\mathrm{T}}$ product (see appendix B). The second term in (4.14) stems from the L_9 $\mathcal{O}(p^4)$ coupling in the Gasser-Leutwyler lagrangian.

Also, at large values of Q^2 the OPE governs an expansion in powers of $1/Q^2$ and yields

$$Q^2 W(Q^2) \approx \frac{F_0^4}{4} + \frac{F_0^4}{3}\frac{\delta_K^2}{Q^2}\left(1 + \frac{\epsilon}{6}\right) + \mathcal{O}\left(\frac{1}{Q^4}\right). \tag{4.15}$$

If we build an interpolator between the two regimes, our work is done. In this respect we recall that, in the large-N_c limit, the coupling L_9 does not run with the scale and the function $W(Q^2)$ is a meromorphic function of the meson masses, i.e. it has poles at the particles' masses, but no cut. For instance, figure 4 shows the pole structure of the function $\Gamma_{\mu\nu\alpha}$ in eq. (4.7). Thus, according to the Mittag-Leffler theorem [29], the function $W(Q^2)$ can be written as an infinite sum of the principal parts at all poles, i.e.

$$W(Q^2) = \sum_{i=1}^{\infty} \frac{r_i}{(Q^2 + M_i^2)^{p_i}}\,, \qquad p_i = 1, 2, 3, \ldots \tag{4.16}$$

Lacking a solution to large-N_c QCD, the value of the masses, M_i, and the residues, r_i, remain unknown. However, using the constraints at low and large Q^2 in eqs. (4.14), (4.15)

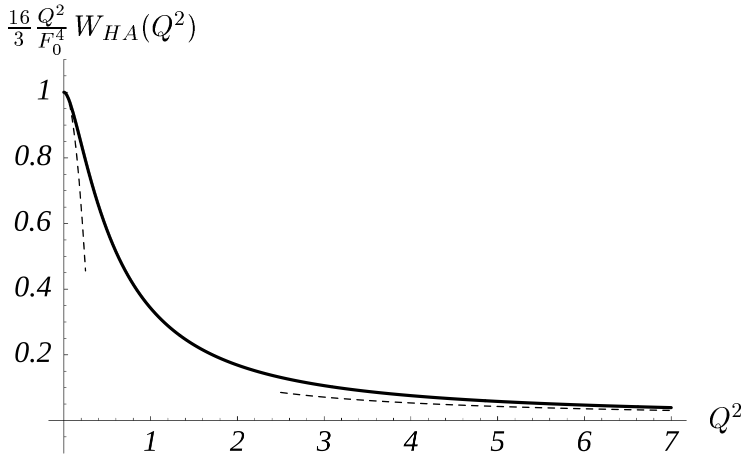

Figure 5: Profile of the function $Q^2\ W_{HA}(Q^2)$ together with the chiral and OPE behavior at low and high Q^2 (in $D = 4$), respectively. The pion contribution has been subtracted (the "a" term in eq. (4.17)), according to the discussion in the text.

one may build a rational interpolator to the function $W(Q^2)$. This interpolator consists in restricting the sum in eq. (4.16) to a *finite* sum, adjusting the residues so that they match the two expansions given by (4.14), (4.15). The rational interpolator so constructed constitutes an approximation to large-N_c QCD [14, 15]. The simplest of such interpolators is given by

$$Q^2 W(Q^2)_{HA} = a + \frac{A}{Q^2 + M_V^2} + \frac{B}{(Q^2 + M_V^2)^2} + \frac{C}{(Q^2 + M_V^2)^3} + \frac{D}{(Q^2 + M_V^2)^4}\,, \qquad (4.17)$$

where we shall take $M_V \simeq 0.77\,\mathrm{GeV}$ as an estimate of M_ρ in the chiral and large-N_c limits. This way one may determine the 5 unknowns a, A, B, C and D from the 3 conditions from the chiral expansion (4.14) and the 2 conditions from the OPE (4.15) as the solutions to the *algebraic* equations:

$$\begin{aligned}
\frac{A}{M_V^2} + \frac{B}{M_V^4} + \frac{C}{M_V^6} + \frac{D}{M_V^8} &= \frac{7}{16}F_0^4 - a \\
\frac{A}{M_V^4} + \frac{2B}{M_V^6} + \frac{3C}{M_V^8} + \frac{4D}{M_V^{10}} &= 0 \\
\frac{A}{M_V^6} + \frac{3B}{M_V^8} + \frac{6C}{M_V^{10}} + \frac{10D}{M_V^{12}} &= -\frac{3}{2}\frac{N_c^2}{(24\pi^2)^2} + 3L_9^2 \\
a &= \frac{F_0^4}{4} \\
A &= \frac{F_0^4\delta_K^2}{3}\left(1 + \frac{\epsilon}{6}\right).
\end{aligned} \qquad (4.18)$$

Notice that, since we use dimensional regularization, the pion contribution represented by the a term in eq. (4.17) gives a vanishing contribution to the integral (4.6), although it does contribute to the chiral and OPE constraints in eqs. (4.18). This is why the resulting function $Q^2W_{HA}(Q^2)$, together with the OPE and chiral extrapolations, are being plotted in figure 1 with the pion contribution subtracted.

Equating expressions (4.5) and (4.6), using (4.17) in order to approximate the integral over Q^2, one obtains the contribution from the effective lagrangian (3.10) to the coupling constant $\Lambda^2_{S=2}$ in the weak chiral lagrangian (4.1) as

$$\Lambda^2_{S=2}|_{\mathrm{dim-8}} = \frac{4}{3}\delta_K^2 \left\{ \lambda_u^2 \left(\log \frac{\mu^2}{M_V^2} - \frac{1}{3} \right) - c_8(\mu) \right\} + 4\frac{\lambda_u^2}{F_0^4} \left[\frac{B}{M_V^2} + \frac{C}{2M_V^4} + \frac{D}{3M_V^6} \right] \quad (4.19)$$

As one can see, the constraint of the OPE in the last equation in (4.18) enforces the cancelation of the μ dependence from the Wilson coefficient $c_8(\mu)$, eq. (3.9), with that coming from the integration of meson resonances, thereby obtaining explicit matching between short and long distances.

The cancelation of the μ dependence agrees with that found in ref. [13] in terms of quark diagrams. However, ref. [13] did not consider the constraints imposed by the chiral behavior in eq. (4.14) and, as a consequence, the result depended on an ad-hoc infrared regulator with the ensuing uncertainty in the result.

Using $L_9 = 7 \times 10^{-3}$, $F_0 = 0.087\,\mathrm{GeV}$, $M_V = 0.77\,\mathrm{GeV}$, as an estimate of the vector resonance mass in the chiral and large-N_c limits, $\delta_K^2 = 0.12 \pm 0.07\,\mathrm{GeV}^2$ [30][10] and the charm mass $m_c(m_c) = 1.3\,\mathrm{GeV}$, we find that

$$\Lambda^2_{S=2}|_{\mathrm{dim-8}} = \left\{ \frac{26}{9}\lambda_c\lambda_u\delta_K^2 + \lambda_u^2 (0.95\,\mathrm{GeV})^2 \right\} (1 \pm 0.3)\,, \quad (4.20)$$

where the error is an estimate of the size of a typical $1/N_c$ correction.[11] Therefore, as we discussed in the introduction, $\mathrm{Re}\,\Lambda^2_{S=2}|_{\mathrm{dim-8}} \sim \lambda_u^2 \Lambda^2_{\mathrm{QCD}}$ with $\Lambda_{\mathrm{QCD}} \sim 1\,\mathrm{GeV}$, whereas $\mathrm{Im}\,\Lambda^2_{S=2}|_{\mathrm{dim-8}} \sim \mathrm{Im}(\lambda_c\lambda_u)\delta_K^2$ with $\delta_K \sim 350\,\mathrm{MeV}$.

5. Results

Armed with the lagrangian (4.1), the dimension-six contribution (4.2), and our result for the dimension-eight contribution (4.19), one can now calculate the matrix element $\langle K^0 | \mathcal{H}^{S=2}_{\mathrm{eff}} | \overline{K}^0 \rangle$.

In the case of ϵ_K, collecting all the terms, we find

$$\begin{aligned} \epsilon_K = {} & e^{i\pi/4} \frac{G_F^2 F_0^2 M_K}{6\sqrt{2}\pi^2 \left(M_{K_L} - M_{K_S}\right)} \times \qquad (5.1) \\ & \times \mathrm{Im} \left\{ \frac{3}{4} g_{S=2} \left[\eta_1 \lambda_c^{*2} m_c^2 + \eta_2 \lambda_t^{*2} \left(m_t^2\right)_{\mathrm{eff}} + 2\eta_3 \lambda_c^* \lambda_t^* m_c^2 \log \frac{m_t^2}{m_c^2} \right] + \frac{3}{4} \Lambda^{*2}_{S=2} \Big|_{\mathrm{dim\text{-}8}} \right\}, \end{aligned}$$

[10] Strictly speaking, the operator on the left-hand side of (4.10) runs with the scale μ. However, this amounts to only a 10% change if μ is varied in the range $1 - 2\,\mathrm{GeV}$, and we shall neglect it.

[11] One can make a naive check by building another interpolator imposing the extra condition of having exact agreement with the OPE curve at, e.g., $Q^2 = 5\,\mathrm{GeV}^2$, which seems to be large enough for the OPE to be trusted. This requires the introduction of an extra pole which we took to be a single pole located at the A1 mass. The result obtained in this case differed from the central value in (4.20) by $\sim 10\%$.

whereas for the Kaon mass difference, one has that

$$M_{K_L}-M_{K_S}=\frac{1}{M_K}\,\mathrm{Re}\langle K^0|\mathcal{H}_{\mathrm{eff}}^{S=2}(0)|\overline{K}^0\rangle=\frac{G_F^2}{3\pi^2}F_0^2M_K\times \tag{5.2}$$
$$\times\,\mathrm{Re}\left\{\frac{3}{4}g_{S=2}\left[\eta_1\lambda_c^{*2}m_c^2+\eta_2\lambda_t^{*2}\left(m_t^2\right)_{\mathrm{eff}}+2\eta_3\lambda_c^*\lambda_t^*m_c^2\log\frac{m_t^2}{m_c^2}\right]+\frac{3}{4}\Lambda_{S=2}^{*2}|_{\text{dim-8}}\right\}.$$

In these two expressions $\Lambda_{S=2}^2|_{\text{dim-8}}$ is given by eq. (4.20). We emphasize that these results are still in the chiral limit. We also remind the reader that, in the large-N_c limit, $g_{S=2}=1$. Nevertheless, some chiral and $1/N_c$ corrections to the contribution of dimension-six operators are known analytically. They amount to the replacement $g_{S=2}\to\frac{4}{3}\widehat{B}_K^\chi\frac{F_K^2}{F_0^2}$, where $\widehat{B}_K^\chi=0.36\pm0.15$ [21] and $F_K=0.114\,\mathrm{GeV}$. Lattice calculations at the kaon mass modify the above result by further changing $\widehat{B}_K^\chi\to\widehat{B}_K$, with $\widehat{B}_K\simeq0.86\pm0.15$ [31, 32].

After all these replacements are made, and feeding these expressions with $m_c=1.3\,\mathrm{GeV}$, $m_t=175\,\mathrm{GeV}$, $M_K=0.498\,\mathrm{GeV}$ and the central values of the CKM matrix entries given by ref. [7], we find that the $\Lambda_{S=2}^2|_{\text{dim-8}}$ contribution is a -5% correction to the $\eta_1\lambda_c^{*2}m_c^2$ term in ϵ_K, which turns into a $+0.5\%$ correction to ϵ_K itself. If the comparison was made in the strict chiral and large-N_c limits, the latter correction would become $+1\%$.

Concerning the kaon mass difference, we find that the $\Lambda_{S=2}^2|_{\text{dim-8}}$ contribution becomes a $+10\%$ correction to the dimension-six result. This relative correction becomes $+25\%$ in the strict chiral and large-N_c limits.

We emphasize that the difference in size between the corrections to ϵ_K and the kaon mass difference is mainly due to the fact that $\delta_K\sim350\,\mathrm{MeV}$ whereas $\Lambda_{\mathrm{QCD}}\sim1\,\mathrm{GeV}$ is much larger (see eq. (4.20)).

Using eq. (5.2), we find $M_{K_L}-M_{K_S}|_{\mathrm{SD}}=(3.1\pm0.5)\times10^{-15}\,\mathrm{GeV}$. Since, experimentally, we know that $M_{K_L}-M_{K_S}=(3.490\pm0.006)\times10^{-15}\,\mathrm{GeV}$ [33], we conclude that short distances amount to $(90\pm15)\%$ of this mass difference when the value $\widehat{B}_K\simeq0.86\pm0.15$ is used. The rest presumably must come from the double insertion of the $\Delta S=1$ chiral lagrangian at $\mathcal{O}(p^4)$, but this calculation is beyond the scope of the present paper.

Acknowledgments

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A. The Schwinger operator formalism

We performed the calculation in the Schwinger operator formalism [27]. This formalism is a very convenient method to do calculations involving covariant derivatives because it preserves gauge invariance at all stages of the calculation. This is unlike ordinary diagrammatic perturbation theory where the ∂_μ and the g_sG_μ sitting in a covariant derivative appear at different orders in the expansion. At the same time we shall regulate divergent integral with dimensional regularization by dimensional reduction [34]. In this regularization one keeps the algebra of Dirac matrices in four dimensions but the momentum integrals are kept D dimensional.

One starts by defining quantum mechanical operators $\widehat{X}_\mu, \widehat{P}_\nu$ satisfying the eigenvalue equation

$$\widehat{X}_\mu |x\rangle = x_\mu |x\rangle\,, \tag{A.1}$$

and the commutation relations

$$\begin{aligned} \left[\widehat{X}_\mu, \widehat{P}_\nu\right] &= -ig_{\mu\nu} \\ \left[\widehat{P}_\mu, \widehat{P}_\nu\right] &= ig_s \frac{\lambda^a}{2} G^a_{\mu\nu}\,, \end{aligned} \tag{A.2}$$

where λ^a are color Gell-Mann matrices satisfying $\mathrm{Tr}\,\lambda^a\lambda^b = 2\delta^{ab}$ and $\widehat{P}_\mu = i\partial_\mu + g_s\frac{\lambda^a}{2}G^a_\mu$ in position space. One has that

$$\langle x|\widehat{\not{P}}\,|y\rangle = i\not{D}_x\ \delta(x-y)\,, \tag{A.3}$$

where the covariant derivative is understood as acting upon the delta function. With this definition the quark propagator can be expressed as:

$$S(k) = \int d^4x e^{ik\cdot x}\langle x|\frac{1}{\widehat{\not{P}}-m_i}|0\rangle = \int d^4x\langle x|\frac{1}{\widehat{\not{P}}+\not{k}-m_i}|0\rangle \tag{A.4}$$

where the second equality follows from the relation

$$e^{ik\cdot\widehat{X}}\widehat{P}_\nu = (\widehat{P}_\nu + k_\nu)e^{ik\cdot\widehat{X}}\,. \tag{A.5}$$

With these definitions we can apply the formalism to $\Delta S = 2$ transitions. Any quark field $q(x)$ is to be understood as $q(\widehat{X})$ so that one has $q(\widehat{X})|y\rangle = q(y)|y\rangle$. For example, let us look at the diagram depicted in figure 3. The procedure can be straightforwardly extended to the remaining ones. The contribution from this diagram can be obtained in coordinate space from the following expression:

$$\begin{aligned} &\frac{G_F^2}{2}M_W^4\lambda_u^2\int d^4x\,d^4y\,d^4z\,d^4w\ \langle x|\bar{s}(1+\gamma_5)\gamma^\mu\frac{1}{\widehat{\not{P}}+\not{k}-m_u}\gamma^\nu(1-\gamma_5)d|y\rangle\ \times \\ &\times\langle x|\frac{1}{k^2-M_W^2}|z\rangle\langle w|\bar{s}(1+\gamma_5)\gamma_\nu\frac{1}{\widehat{\not{P}}+\not{k}-m_u}\gamma_\mu(1-\gamma_5)d|z\rangle\ \langle y|\frac{1}{k^2-M_W^2}|w\rangle\,. \end{aligned} \tag{A.6}$$

The integration of the W particle amounts to the expansion:

$$\langle x|\frac{1}{k^2-M_W^2}|z\rangle = \frac{-1}{M_W^2}\ \delta(x-z)+\cdots \tag{A.7}$$

This has to be done twice. One of the Dirac deltas can be used to reduce the number of integrals, whereas for the second one we use its representation in momentum space:

$$\delta(x-y) = \int\frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)} \tag{A.8}$$

which will eventually give rise to the loop integral.

Now we can expand the quark propagators to any order in the soft momentum $\widehat{\not P}$ in the following way:

$$\frac{1}{\widehat{\not P}+\not k-m_u}=\frac{1}{\not k-m_u}-\frac{1}{\not k-m_u}\widehat{\not P}\frac{1}{\not k-m_u}+\frac{1}{\not k-m_u}\widehat{\not P}\frac{1}{\not k-m_u}\widehat{\not P}\frac{1}{\not k-m_u}+\cdots . \tag{A.9}$$

Keeping terms of $\mathcal{O}(P^2)$, since we are interested in dimension-eight operators, and after some diracology, we end up with:

$$\begin{aligned}\langle x|P_L\frac{1}{\widehat{\not P}+\not k-m_u}P_L|0\rangle &= \langle x|P_L\frac{\not k}{k^2-m_u^2}P_L|0\rangle+\langle x|P_L\left[\frac{\widehat{\not P}}{k^2-m_u^2}-\frac{2(\widehat{P}\cdot k)\not k}{(k^2-m_u^2)^2}\right]P_L|0\rangle+\\ &+\langle x|P_L\left[\frac{4\not k\,(\widehat{P}\cdot k)^2}{(k^2-m_u^2)^3}-\frac{\widehat{P}^2\not k}{(k^2-m_u^2)^2}-\frac{(\widehat{P}\cdot k)\widehat{\not P}}{(k^2-m_u^2)^2}-\right.\\ &\left.-\frac{\widehat{\not P}\,(\widehat{P}\cdot k)}{(k^2-m_u^2)^2}-i\frac{\epsilon^{\alpha\beta\gamma\delta}\widehat{P}_\alpha\widehat{P}_\beta k_\gamma\gamma_\delta\gamma_5}{(k^2-m_u^2)^2}\right]P_L|0\rangle\end{aligned} \tag{A.10}$$

which is the expression to be employed for the u quark propagator hereafter.

Now we use (A.3) to convert $\widehat{P}_\mu$ operators into covariant derivatives. Integration by parts shifts the derivatives to the quark fields and delta functions can then be integrated in a trivial way. Thus, one can write, for instance:

$$\begin{aligned}\int\frac{d^Dk}{(2\pi)^D}\int d^4x\,d^4y\,\bar{s}_L(x)\gamma^\mu\langle x|\frac{\not k}{k^2-m_u^2}|0\rangle\gamma^\nu d_L(0)\bar{s}_L(y)\gamma_\nu\langle y|\frac{\widehat{P}^2\not k}{(k^2-m_u^2)^2}|0\rangle\gamma_\mu d_L(0) &=\\ =\bar{s}_L(0)\overleftarrow{D}^2\gamma_\nu d_L(0)\bar{s}_L(0)\gamma_\mu d_L(0)\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu}{(k^2-m_u^2)^3}&\\ =g_s\bar{s}_L(0)\tilde{G}_{\nu\sigma}\gamma^\sigma d_L(0)\bar{s}_L(0)\gamma_\mu d_L(0)\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu}{(k^2-m_u^2)^3}&\end{aligned} \tag{A.11}$$

where the Dirac structure has already been simplified and the operator in the second line has been rewritten in terms of the dual

$$\tilde{G}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}G_{\alpha\beta},\qquad G_{\mu\nu}=\frac{\lambda^a}{2}G^a_{\mu\nu},\qquad \varepsilon^{0123}=1\,. \tag{A.12}$$

In order to get to the result in eq. (A.11) one may use the identity $\overleftarrow{D}^2=\frac{g_s}{2}\sigma_{\mu\nu}G^{\mu\nu}+\overleftarrow{\not D}^{\,2}$, and the equations of motion for the quark fields in the chiral limit. Furthermore, using

$$\bar{s}_LG^{\alpha\nu}\gamma_\nu d_L=\frac{-i}{g}\left\{\bar{s}_L\left(D_\alpha\not D-\overleftarrow{\not D}\,D_\alpha\right)d_L-\partial_\mu\left(\bar{s}_L\gamma_\nu D_\alpha d_L\right)\right\}, \tag{A.13}$$

one finds that

$$\langle\overline{K}^0(p)|\bar{s}_L\gamma^\nu G_{\alpha\nu}d_L|0\rangle=\frac{i}{g}\,\partial_\mu\langle\overline{K}^0(p)|\bar{s}_L\gamma^\mu D_\alpha d_L|0\rangle\sim\mathcal{O}(p^2p_\alpha) \tag{A.14}$$

due to eq. (3.4) and, thus, it can be neglected.

Upon performing the integral over k in eq. (A.11) one can easily extract the μ dependence. After adding up all the contributions, one obtains the result (3.7) in the text.

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B. Ward identities

Let the Green's function $\Gamma_{\mathcal{A}}^{\mu\nu\alpha}(q,p)$ be

$$\Gamma^{\mathcal{A}}_{\mu\nu\alpha}(q,p) \equiv \int d^4x d^4y\ e^{-i(qx+py)} \langle 0|\mathrm{T}\left\{ R^{ds}_{\alpha}(y) L^{su}_{\mu}(x) L^{ud}_{\nu}(0) \right\}|0\rangle\,, \tag{B.1}$$

where in this expression the symbol $\mathrm{T}\{\cdots\}$ stands for the ordinary covariant chronological product. Because of the chiral anomaly one has the following Ward identities:

$$\begin{aligned} q_\mu \Gamma_{\mathcal{A}}^{\mu\nu\alpha}(q,p) &= -i\Pi^{\nu\alpha}_{LR}(p) - i\frac{N_c}{48\pi^2}\varepsilon^{\lambda\sigma\nu\alpha} q_\lambda p_\sigma \\ (q+p)_\nu \Gamma_{\mathcal{A}}^{\mu\nu\alpha}(q,p) &= i\Pi^{\mu\alpha}_{LR}(p) + i\frac{N_c}{48\pi^2}\varepsilon^{\lambda\sigma\alpha\mu} q_\lambda p_\sigma \\ p_\alpha \Gamma_{\mathcal{A}}^{\mu\nu\alpha}(q,p) &= i\frac{N_c}{24\pi^2}\varepsilon^{\mu\nu\lambda\sigma} q_\lambda p_\sigma\,. \end{aligned} \tag{B.2}$$

One can also define a similar Green's function to that in eq. (B.1) but with a different T product, which we shall denote by $\widehat{T}\{\cdots.\}$, which produces the left-right, or factorized, form of the anomaly [23, 24]. In this case the new Green's function, which we shall denote by $\Gamma_{\mu\nu\alpha}(q,p)$, satisfies the naive Ward identities *without* the anomaly terms. In other words, one has that

$$\begin{aligned} q_\mu \Gamma^{\mu\nu\alpha}(q,p) &= -i\Pi^{\nu\alpha}_{LR}(p) \\ (q+p)_\nu \Gamma^{\mu\nu\alpha}(q,p) &= i\ \Pi^{\mu\alpha}_{LR}(p) \\ p_\alpha \Gamma^{\mu\nu\alpha}(q,p) &= 0\,. \end{aligned} \tag{B.3}$$

The relationship between the two Green's functions is given by the equality

$$\Gamma^{\mu\nu\alpha}(q,p) = \Gamma_{\mathcal{A}}^{\mu\nu\alpha}(q,p) + i\frac{N_c}{24\pi^2}\varepsilon^{\mu\nu\alpha\lambda} q_\lambda + i\frac{N_c}{48\pi^2}\varepsilon^{\mu\nu\alpha\lambda} p_\lambda\,. \tag{B.4}$$

As one can see in the equation above, they differ by a local counterterm, i.e. a polynomial in momenta.

For small momentum p, one can write down the most general structure for $\Gamma^{\mu\nu\alpha}(q,p)$ compatible with the conditions in eq. (B.3). Defining, as usual,

$$\Pi^{\nu\alpha}_{LR}(p) \equiv \int d^4x\, e^{-ip\cdot y} \langle 0|T\left\{ R^{\alpha}_{ds}(y) L^{\nu}_{sd}(0)\right\}|0\rangle = \left(g^{\nu\alpha} - \frac{p^\nu p^\alpha}{p^2}\right) \Pi_{LR}(p^2)\,, \tag{B.5}$$

and recalling that

$$\Pi_{LR}(0) = -i\frac{F_0^2}{2}\,, \tag{B.6}$$

we find the following form for $\Gamma^{\mu\nu\alpha}(q,p)$ for small p:

$$\Gamma_{\mu\nu\alpha}(q,p) = \Pi_{LR}(0)\mathcal{T}^{S}_{\mu\nu\alpha} + I_1(Q^2)\mathcal{T}^{ST}_{\mu\nu\alpha} + I_2(Q^2)\mathcal{T}^{A}_{\mu\nu\alpha} + \mathcal{O}(p)\,, \tag{B.7}$$

where $Q^2 = -q^2$, and

$$\begin{aligned}
\mathcal{T}^{S}_{\mu\nu\alpha} &= \frac{i}{p^2 q^2}\left[p^2 q_\mu g_{\alpha\nu} + p^2 q_\nu g_{\alpha\mu} - p_\alpha p_\nu q_\mu - p_\alpha p_\mu q_\nu + q_\mu q_\nu \left(\frac{p\cdot q}{q^2} p_\alpha - \frac{p^2}{q^2} q_\alpha\right)\right] \\
\mathcal{T}^{ST}_{\mu\nu\alpha} &= \left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\left(\frac{p\cdot q}{p^2} p_\alpha - q_\alpha\right) \\
\mathcal{T}^{A}_{\mu\nu\alpha} &= \left[i\varepsilon_{\mu\nu\lambda\sigma} q^\sigma \frac{p_\alpha p^\lambda}{p^2} - i\varepsilon_{\mu\nu\alpha\lambda} q^\lambda\right].
\end{aligned} \tag{B.8}$$

This is the expression (4.11) in the main text. In eq. (B.8), the superindices S, T, A summarize the symmetry properties of each tensor with respect to the exchange of the indices μ, ν: $\mathcal{T}^{S}_{\mu\nu\alpha}$ is symmetric, $\mathcal{T}^{ST}_{\mu\nu\alpha}$ is also symmetric and, furthermore, transverse with respect to q_μ. Finally, $\mathcal{T}^{A}_{\mu\nu\alpha}$ is $\mu\nu$ antisymmetric.

At low momentum one finds that

$$I_1(Q^2) = 2\frac{L_9}{Q^2} + \mathcal{O}(Q^0)\,, \qquad I_2(Q^2) = -\frac{N_c}{24\pi^2} + \mathcal{O}(Q^2)\,, \tag{B.9}$$

whereas, at large momentum one has

$$I_1(Q^2) = \frac{F_0^2}{2Q^4} + \mathcal{O}\left(\frac{1}{Q^6}\right), \qquad I_2(Q^2) = -\frac{F_0^2}{2Q^2} + \mathcal{O}\left(\frac{1}{Q^4}\right). \tag{B.10}$$

As a consequence of the symmetry properties of the tensors $\mathcal{T}_{\mu\nu\alpha}$ there are no crossed terms in the following contraction:

$$\int d\Omega_q \Gamma^{\mu\nu\alpha}(q,p)\Gamma_{\nu\mu\beta}(q,p) = \left(\frac{p^\alpha p_\beta}{p^2} - g^\alpha_\beta\right)\left\{\frac{3}{4}I_1^2(Q^2)Q^6 - \frac{3}{2}I_2^2(Q^2)Q^2 + \frac{7}{16}\frac{F_0^4}{Q^2}\right\} + \mathcal{O}(p^2)\,, \tag{B.11}$$

and eq. (4.12) in the main text follows.

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